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Parametric nonlinear sloshing in a 2D rectangular tank with finite liquid depth

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ABSTRACT: An investigation of parametrically-excited sloshing in a two-dimensional (2D) rectangular tank with finite liquid depth is described. The analysis is based on the adaptive multimodal approach of nonlinear sloshing in a rectangular tank, described in Faltinsen and Timokha (2001). Following the standard adaptive mode ordering, a finite-dimensional system of ordinary differential equations is obtained. Third-order polynomial nonlinearities are retained. The external vertical forcing of the tank is assumed to be of "sufficiently" small amplitude. The novel part of the work lies in the advanced investigation of the nonlinear free surface oscillations that brought to light a new (in parameters space) region of liquid surface instability that was up to now considered as quiescent.

1 INTRODUCTION

Parametrically excited engineering systems can often exhibit dangerous behaviour (Ibrahim 1985). Parametric sloshing is the motion of a liquid's free surface due to an excitation perpendicular to the plane of the undisturbed free surface. Such vertical excitation of ship tanks could be produced by the heaving motion in a seaway. In our case the heave is considered as a harmonic function. This arises physically in combination with excitations in other modes of ship motion; for example in pitch and in roll. In the current work we have considered however the heave excitation alone in order to clarify the free surface dynamics under pure parametric excitation. The issue is not new, yet it is less often considered as important compared to cases of directly excited sloshing (Dodge 1966). In an investigation of parametric sloshing the nonlinear treatment of the free surface is very essential; otherwise unrealistic infinite free surface displacements will be uniformly predicted inside the instability region.

A concise overview of research concerning the nonlinear behaviour of liquids contained in tanks of various shapes and subjected to parametric excitation can be found for example, in the book of Ibrahim (2005). Standing waves generated in vertically oscillating tanks were firstly studied experimentally by Faraday (1831), Mathiessen (1868, 1870) and Lord Rayleigh (1883a & b, 1887). The same problem was investigated theoretically by Lewis (1950), Taylor (1950), Benjamin & Ursell (1954), Konstantinov et al. (1978), Nevolin (1985), Feng & Sethna (1989), Simoneli & Gollub (1989), Henderson & Miles (1990), Nagata (1991), Miles (1994), Perlin & Schultz (2000). Different numerical approaches to sloshing prediction have been reported by Telste (1985), Chen et al. (1996), Takizawa & Kondo (1995), Chern et al. (1999), Pawell (1997), Turnbull et al. (2003), Wu et al. (1998, 2001 and 2007), Fradnsen (2003), Y. Kim et al. (2001, 2007) treating the moving free surface either by using Lagrangian tracking of free surface nodes with regrinding; or by mapping. Both have advantages and disadvantages; however a common drawback is that they are not the most appropriate for long time simulations.

In the current work we have adopted a well-known semi-analytical method developed by Faltinsen and Timokha (2001) that is based on modal analysis. However, as our objective here is not to advance the modelling of sloshing but to deepen into the character of the nonlinear oscillations exhibited by the free surface of a liquid in a two-dimensional (2D) rectangular tank with finite liquid depth, a direct method for capturing nonlinear steady dynamics is coupled to the hydrodynamic model. More specifically, we couple the model with a "continuation analysis" algorithm of nonlinear dynamics in order to predict (in a single run) the amplitudes of steady liquid surface oscillations as the frequency and/or the amplitude of excitation are varied, without performing multiple simulations (which would not capture the unstable oscillations anyway). The focus is on determining the region of parameters' values where free surface activity takes place (termed here as "region of instability"). For the model that we use, this region of instability is predicted to be more extended than the one associated with linear parametrically excited systems (as described for example in Benjamin & Ursell 1954). The unique feature of the newly discovered region is that it is initial-conditions-dependent i.e., one may obtain a stable surface wave or a flat surface depending on the exact state of the free surface when the harmonic excitation was firstly applied.

2 FORMULATION OF THE PROBLEM

We consider a mobile, rectangular, smooth and rigid tank, filled partly by an inviscid, incompressible fluid. Liquid depth is finite; but the tank top is high enough so that it is never reached by the moving liquid. The flow is two-dimensional and irrotational. The origin of the coordinate system is placed at the middle of the mean free surface (Figure 1).

The problem of sloshing of an incompressible fluid with irrotational flow when part of the boundary (the free surface) is free to move is formulated in the standard manner in terms of the Laplace equation in the fluid, with suitable boundary conditions:

$$\Delta\Phi = 0 \text{ in } Q(t),$$
$$\frac{\partial\Phi}{\partial n} = \vec{u}_0 \cdot \vec{n} \text{ on } S(t),$$
$$\frac{\partial\Phi}{\partial n} = \vec{u}_0 \cdot \vec{n} - \frac{\partial Z/\partial t}{|\nabla Z|} \text{ and}$$
$$\frac{\partial\Phi}{\partial t} - \nabla\Phi\cdot\vec{u}_0 + \frac{1}{2}(\nabla\Phi)^2 + U_g = 0 \text{ on } \Sigma(t) \quad (1)$$

$Q(t)$ is the fluid volume, $\Sigma(t)$ is the free surface which is associated with the equation $Z(y, z, t) = 0$, $S(t)$ is the tank surface below $\Sigma(t)$, t is time, $\Phi(y, z, t)$ is the velocity potential in the reference frame, and $\vec{n}$ is the unit vector that is normal to $S(t)$. Tank's velocity is $\vec{u}_0(t) = \dot{n}_3 \cdot \vec{e}_3$, where n_3 is the magnitude of external excitation and $\vec{e}_3$ is the unit vector in the z- axis.

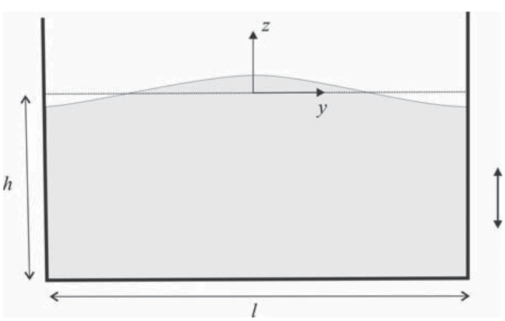

Figure 1. The origin of the coordinate system is placed at the middle of the mean free surface. Tank length and height are denoted by *l, h* respectively. Tank is free to move in the z-axis.

3 NONLINEAR ASYMPTOTIC ADAPTIVE MODAL ANALYSIS

The multimodal method uses a Fourier series representation of the solution with time-dependent unknown coefficients. The sloshing problem is expressed by means of two functions, describing the free-surface elevation and the velocity potential. Faltinsen et al. (2000) postulated their Fourier series representations as follows:

$$\zeta(y,t) = \sum_{i=1}^{\infty} \beta_i(t) f_i(y),$$
$$\Phi(y,z,t) = \sum_{i=1}^{\infty} R_i(t)\varphi_i(y,z) \quad (2)$$

The modal representation (2) is based upon the functions $\{f_i\}$ and $\{\varphi_i\}$ which must provide "complete" sets on the mean free surface and the whole tank domain, respectively. The most common choice for the basis $\{\varphi_i\}$ is the set of linear natural modes. However, the natural modes are theoretically defined only in the unperturbed hydrostatic domain. In view of this problem, we interpret the natural modes as an *asymptotic* basis, assuming that the free surface is, to some extent, asymptotically close to its unperturbed state (Faltinsen & Timokha, 2009). In our case the modal basis $f_i(y)$ and the set of functions $\varphi_i(y, z)$ coincide with the linear natural sloshing modes (3) as derived by Faltinsen & Timokha (2002).

$$\varphi_i(y,z) = \cos\left(\frac{\pi i}{l}\left(y + \frac{1}{2}l\right)\right) \times \frac{\cosh(\frac{\pi i(z)}{l})}{\cosh(\frac{\pi i h}{l})}$$
$$f_i(y) = \cos\frac{\pi i\left(y + \frac{1}{2}l\right)}{l}, \quad i \geq 1 \quad (3)$$

By using the asymptotic non-linear modal theory we obtain an infinite-dimensional system of nonlinear differential equations (modal system). Following the adaptive approach proposed by Faltinsen & Timokha (2001), this modal system could be asymptotically reduced to an infinite-dimensional system of ODEs. An important fact is that asymptotically truncated systems may use natural modes because the procedure needs only the completeness of $\varphi_n(y, z)$ in the unperturbed liquid domain (Faltinsen & Timokha, 2002). If, in the first instance, nonlinear terms are kept only up to third-order, the considered system comes to the following form in the case of vertical excitation:

$$\sum_{i=1}^{p} \ddot{\beta}_i (\delta_{p\mu} + d_{p,q}^{1,\mu} \sum_{i=1}^{q} \beta_i + d_{p,q,r}^{2,\mu} \sum_{i=1}^{q} \beta_i \sum_{i=1}^{r} \beta_i) +$$

$$\sum_{i=1}^{p} \dot{\beta}_i \sum_{i=1}^{q} \dot{\beta}_i (t_{p,q}^{0,\mu}) + \sum_{i=1}^{p} \dot{\beta}_i \sum_{i=1}^{q} \beta_i \sum_{i=1}^{r} \beta_i (t_{p,q,r}^{1,\mu}) +$$

$$[\sigma_\mu^2 + \frac{\ddot{n}_3}{l} \pi\mu \tanh(\frac{\pi\mu h}{l})]\beta_\mu = 0, \ \mu = 1, 2 \ldots \quad (4)$$

where δ stands for Kronecker's delta and p, q, r are upper summation limits. The d and t coefficients can be expressed as functions of the ratio of liquid depth to tank breadth (such analytical expressions can be found in Faltinsen and Timokha 2001). σ_μ represents the μ^{th} natural frequency and is given (Faltinsen & Timokha, 2009) by the equation:

$$\sigma_\mu = \sqrt{g\left(\frac{\pi\mu}{l}\right)\tanh\left(\frac{\pi\mu}{lh}\right)}, \ \mu = 1, 2 \ldots \quad (5)$$

Furthermore, by using the condition $\beta_n = O(\varepsilon^{1/3})$, $\beta_i = O(\varepsilon)$, $i \geq n + 1$ we could introduce more than one dominant mode (in contrast with a Moiseev-like ordering). This leads to a finite-dimensional nonlinear modal system that will be called from here on "Model-k", where the integer k denotes the number of dominant modes. The coefficient $\varepsilon = n_{3\alpha}/l$ is an indicator of the smallness of excitation. Here it is assumed that $\varepsilon << 1$.

4 LINEAR MODEL

Linearization of the modal system (Equation 4) leads to the following set of uncoupled modal equations:

$$\ddot{\beta}_\mu + \left[\sigma_\mu^2 + \ddot{n}_3 l^{-1} \pi\mu \tanh\left(\frac{\pi\mu h}{l}\right)\right]\beta_\mu = 0, \mu = 1, \ldots \quad (6)$$

By assuming harmonic excitation $n_3 = n_{3\alpha}\cos(\sigma t)$, Equation 6 comes to the form of a set of Mathieu-type equations. Let us now restrict our investigation to the case of vertical excitation with relatively small amplitude, with excitation frequency in the vicinity of the principal parametric resonance of the first mode; i.e. $(\sigma_1/\sigma)^2 \approx 1/4$.

We also choose the tank's height-to-breadth-ratio to be larger than the critical depth ($h/l = 0.03368$) restricting our investigation to *finite liquid depth*. We do this because, according to Fulzt (1962), in the vicinity of critical depth strong changes and amplifications in the liquid behaviour occur.

It is remarked that Faltinsen and Timokha (2009) have introduced to Equation 6 an empirical linear damping term, represented by the damping ratio ζ_1. From a physical perspective, this damping term could empirically account for boundary–layer damping. Incorporating such damping, the linear modal equation for the dominant mode β_1 obtains the following form:

$$\ddot{\beta}_1 + 2\sigma_1\zeta_1\dot{\beta}_1 + \sigma_1^2\left[1 - \frac{n_{3a}\sigma^2}{g} \cdot \cos(\sigma \cdot t)\right]\beta_1 = 0 \quad (7)$$

Benjamin and Ursell (1954) had investigated the free surface elevation under similar forcing, using Mathieu functions for expressing analytically the solution. As well known, depending on parameters' values, a system described by Equation 7 could exhibit stable as well as unstable behaviour.

Frandsen (2004) considered the case of a rectangular tank under vertical forcing and checked the stability of the free surface by using two-dimensional CFD simulations. She has shown that the prediction of the stable regions is in good agreement with Benjamin and Ursell's (1954) predictions when the forcing parameter is small. If the excitation amplitude is raised, nonlinearities due to intermodal interaction have to be considered.

In Figure 2 is shown the well-known stability chart that corresponds to Equation (7), for damping ratio $\zeta_1 = 0.02$. The linear model entails that, to the interior of the curve a solution is unstable and diverges to infinity; whereas every point outside the curve corresponds to a stable steady solution, in the sense of having the liquid surface maintained flat and horizontal.

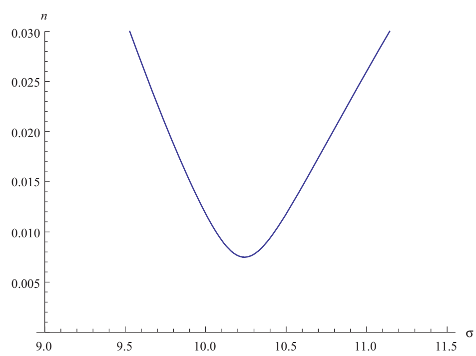

Figure 2. Stability map for the Mathieu type Equation 7, for damping ratio $\zeta_l = 0.02$.

As realised, β_1 stands for the time-dependant response of the free surface. In what follows we track the elevation of free surface at $y = -l/2$; i.e., at its interface with the left tank wall. To obtain the elevation for all other points (y, 0) of the liquid surface β_1 should be multiplied by $\cos(\pi(y + l/2)/l)$.

5 NON-LINEAR MODEL (MODEL-1)

For mode ordering $\beta_1 = O(\varepsilon^{1/3})$, $\beta_\mu = O(\varepsilon)$, $\mu > 1$, and by retaining only β_1 following the same thinking like before, the system of Equations 4 generates the following form ("Model-1"):

$$\ddot{\beta}_1 + 2\zeta_1\sigma_1\dot{\beta}_1 + (\sigma_1^2\beta_1 + Q_1\ddot{n}_3\beta_1) + d_2(\ddot{\beta}_1\beta_1^2 + \dot{\beta}_1^2\beta_1) = 0 \quad (8)$$

where: $d_2 = \dfrac{\pi^2}{4}\left[1 - 2\cdot\tanh(\pi\dfrac{h}{l})\cdot\tanh(2\pi\dfrac{h}{l})\right]$ and

$$Q_1 = \pi\tanh(\frac{\pi h}{l}).$$

As in the linear model case, a damping term has been introduced. Assuming harmonic external excitation $n_3 = n_{3\alpha}\cos(\sigma t)$ with small amplitude, Equation (8) becomes:

$$\ddot{\beta}_1 + 2\zeta_1\sigma_1\dot{\beta}_1 + \sigma_1^2(1 - \frac{n_{3a}\sigma^2}{g}\cdot\cos(\sigma\cdot t))\beta_1 + d_2(\ddot{\beta}_1\beta_1^2 + \dot{\beta}_1^2\beta_1) = 0 \quad (9)$$

As observed, the above non-linear system is the same as the linear plus two non-linear terms that feature products of the surface elevation with the corresponding acceleration and with the corresponding velocity. This makes this problem somehow different from several others that are also modeled through a Mathieu-type equation [e.g., for ships' parametric rolling investigations we usually include nonlinearity only n the stiffness term, Spyrou et al. (2008). It is remarked that all other nonlinear terms of Equation (4) are not present because their coefficients, according to the current approximation, are equal to zero. The only non-zero nonlinear term parameter is d_2.

The nonlinear model of Equation (9) was derived by following the adaptive approach. With the restrictions described above, the same equation could have been produced from the single dominant mode approach, also introduced by Faltinsen et al. (2000). If large-amplitude response occurs; or if the tank-height-to depth-ratio is equal or smaller than the critical depth; or lastly if secondary resonance occurs, the assumption of lowest order dominant mode collapses. In such cases, the adaptive approach is the way to follow for model derivation (Faltinsen and Timokha 2002). We have thus selected to follow the adaptive method, in order to maintain the prospect of comparison of our results against those produced from higher order models that we are concurrently investigating (not included in this paper).

6 CONTINUATION ANALYSIS FOR THE NON-LINEAR MODEL

"Continuation" algorithms usually accept the mathematical model in the so called "autonomous canonical form":

$$\frac{dx}{dt} = f(x;b) \quad (10)$$

Here x and b are, respectively, the state and control parameters' vectors of our problem. Variation of one or more components of the control vector b creates, through solution of the above vector differential equation, branches of steady-state (in our case periodic) solutions. These branches constitute the "spine" of the dynamical response of the system. It is thus imperative to be able to trace such branches of steady-states efficiently, even when nonlinearities are strong and multiplicity of steady solutions arises. For such types of investigation "continuation" is a truly indispensable tool. The specific algorithm implemented here is "MATCONT". For mathematical details Dhooge et al. (2003) can be consulted.

The basic mathematical model expressed through Equation (8) is characterised by explicit time-dependence in the restoring term. Thus, it is not in the autonomous form of Equation (10) entailed by the continuation algorithm. To overcome this, a suitable additional pair of differential equations is introduced with respect to the dummy variables $x = \sin(\sigma t)$, $y = \cos(\sigma t)$. Thereafter, Equation (9) can be converted into the following system of four 1st-order ordinary differential equations that, whilst being equivalent to Equation (10), is not characterised by explicit time-dependence (Spyrou & Tigkas 2011):

$$\begin{aligned}
\dot{\beta}_1 &= \phi_1 \\
\dot{\phi}_1 &= (-2\zeta_1\sigma_1\phi - \sigma_1^2(1 - \frac{n_{3a}\sigma^2}{g}\cdot y)\beta_1 - \phi_1{}^2\beta_1)/(1 + d_2\beta_1{}^2) \\
\dot{x} &= x + \sigma\cdot y - x\cdot(x^2 + y^2) \\
\dot{y} &= y - \sigma\cdot x - y\cdot(x^2 + y^2)
\end{aligned} \quad (11)$$

Due to the cubic nonlinearity of Model-1, inside the unstable region one should expect a stable limit cycle. Selecting such limit cycle as initial state (this is easily captured through simulation) we can use the continuation method to produce the dependence of that "cycle" from the excitation amplitude and/or frequency, while restricting ourselves to the region of small amplitudes so that we don't violate the assumption of relatively small surface elevation that is intrinscic to the model. Such a continuation analysis result is presented in Figure 3. It is observed that there is an excitation region where two different limit cycles coexist. The limits of this region is determined by two bifurcation points. When the excitation amplitude increases from zero and until the first bifurcation is met ("limit point of cycles"), there isn't any limit cycle. This is the area where a stable flat condition is possible for the water surface.

As the amplitude continues to increase, two coexisting limit cycles are suddenly met, one stable and one unstable. The stable one gets gradually larger as the excitation is increased. The other limit cycle shrinks till the second bifurcation point (that is of "subcritical" type) where it disappears. On the basis of Figure 3 we have extracted Figure 4 that summarizes the dependence of surface elevation upon the excitation amplitude $n_{3\alpha}$. It is remarked that subcritical bifurcations are associated with hysteretic behaviour as explained next:

As the excitation amplitude increases from the zero value to the bifurcation point A, the liquid-free-surface amplitude will jump in a fast dynamic transition to point C right after A is reached. As the excitation level is further increased, the amplitude also increases monotonically along the solid curve CD. If, on the other hand, the excitation begins to decrease while the liquid free surface is in a state in the neighbourhood of point C, the surface amplitude will decrease along the curve DCB until the point B is reached. Figure 4 has a qualitatively similar form with one by Ibrahim (2005) who had targeted an approximation of the solution according to a perturbation technique.

To investigate the stability of these periodic solutions the formal choice is to perform calculation of Floquet multipliers. But we can also get some strong indication about the stability of periodic solutions by simply selecting some forcing level that lies in between the two Branch points of Cycles (BPC); and then integrating the nonlinear system under different initial conditions. As expected from nonlinear dynamics, one of the limit cycles (the outer) is stable and the other one (the inner) is unstable. In Figure 5 is presented the time history of β_1 for two slightly different initial conditions. Initial elevation $\beta_1 = 0.091$ m (with zero initial vertical velocity of the surface) leads to a stable (zero) point; i.e., there is attraction towards the stationary state.

On the other hand, an initial elevation $\beta_1 = 0.095$ m leads to a stable periodic pattern.

Working in a similar manner for more than one frequency we are able to demonstrate the

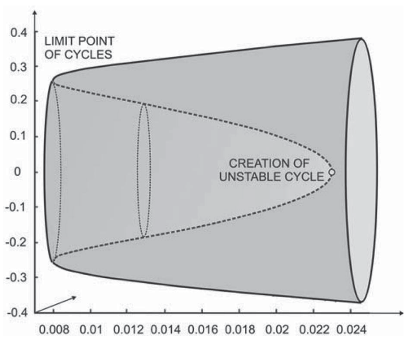

Figure 3. Limit-cycle dependence on excitation amplitude ($n_{3\alpha}$, β_1), for $\sigma = 9.7$ rad/s. There is an excitation region determined by the point of creation of limit cycles and the fold (limit point) of cycles (the two Branch points of Cycles) where two different limit-cycles coexist (one unstable).

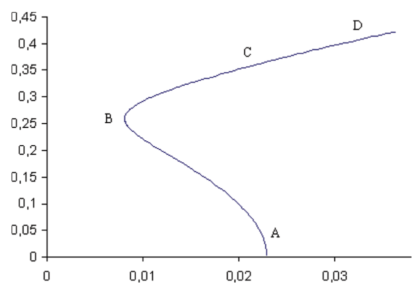

Figure 4. Dependence of the response amplitude (y-axis) on the excitation amplitude $n_{3\alpha}$ (x-axis). Dependence has a qualitatively similar form with that extracted by Ibrahim (2005).

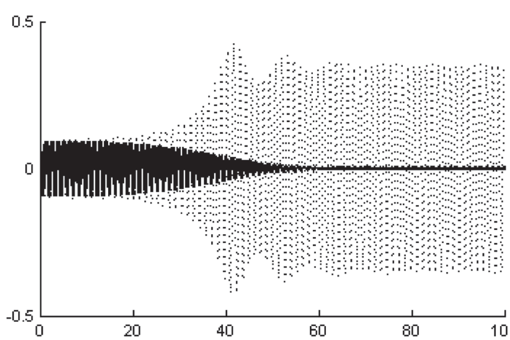

Figure 5. Time history of β_1 for σ = 9.7 rad/s and $n_{3\alpha} = 0.02$. Initial elevation $\beta_1 = 0.091$ m (solid line) leads to a stable (zero) point and $\beta_1 = 0.095$ m (dot line) leads to a stable periodic pattern.

steady dynamic behavior of the autonomous nonlinear system represented by Equation (11), in the frequency region that surrounds $\sigma = 2\sigma_1$. In Figure 6 is summarised this behaviour. As one expects, for small frequency every excitation ends up to a stable point. Increase of the excitation frequency leads, through a fold of limit-cycles bifurcation, to an initial-conditions-dependent area (Figure 6-Left) where a stable and an unstable limit-cycle coexist. Further increase of frequency leads, through the subcritical bifurcation mentioned earlier, to the "classical" area of instability associated with principal parametric resonance (Figure 6-Right), where the unstable limit-cycle disappears and the stable trivial solution becomes unstable. Lastly, a supercritical ("smooth") bifurcation locus represents in fact the boundary with the stability area to the right of the "classical" linear instability area.

Incorporating the above observations into the stability chart of the linear Mathieu-type system we obtain the diagram of Figure 7. If we compare Figure 7 against the similar result of Benjamin and Ursell we see that a new initial-condition-dependent area (area C) has been added. As a result, the forcing-versus-frequency parameters' plane is divided into three areas that are identified in Figure 7. Area A is the stable one where every external excitation (with the fitting specification) leads invariably to a flat liquid surface. Area B is the classical area of instability where external excitation generates periodic oscillations of the free surface. Area C is the initial-condition-dependent area, (in terms of β_1 and φ_1) where the same external

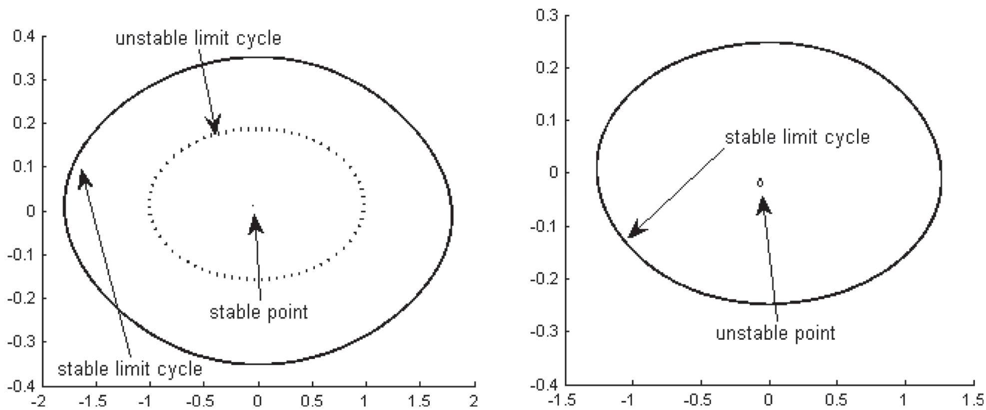

Figure 6. Phase diagram (φ_1, β_1) for different excitation frequencies, $\zeta_1 = 0.02$ m and $n_{3a} = 0.02$ m. Left: initial-condition-dependent area for excitation frequency σ = 9.7 rad/sec. The unstable limit cycle drives the behaviour of the system either to a stable fixed point or to a stable limit cycle, Right: typical behavior in the classical unstable area for excitation frequency σ = 10.2 rad/sec. Every excitation ends to a stable limit-cycle.

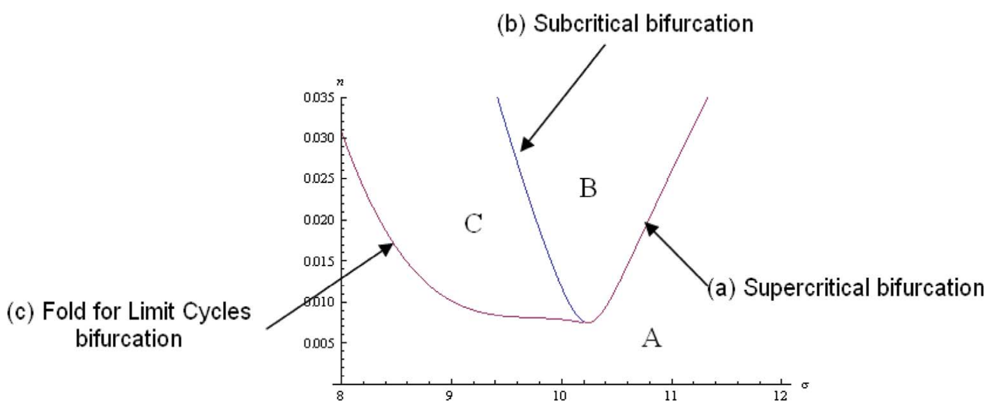

Figure 7. Stability chart for the nonlinear Model-1 system (ζ_1 = 0.02). A new initial-condition-depended area (indicated by C) is added to the linear stability chart. Inside C, one may obtain a stable surface wave or a flat surface depending on how the free surface looked like when the harmonic excitation was firstly applied.

excitation leads either to a quiescent surface or to a wavy one. The phase space structure of the considered dynamical system is quite a common one. The domains of attraction of the two competing stable patterns occupy certain complementary regions of phase space. They are separated by a surface defined by the incoming (stable) manifold of the unstable periodic solution.

7 CONCLUSIONS

2-D liquid sloshing in a rectangular and vertically excited tank has been investigated. A non-linear model has been used, based on modal modelling and according to the adaptive analysis of Faltinsen and Timokha (2001). The work was limited to finite liquid depth, corresponding to a tank-height-to-depth-ratio of 0.4. A global picture of liquid surface dynamics was obtained, befitting to model nonlinearities retained up to third-order. A new area of bi-stability should be added to the stability chart of free surface oscillations. Investigation of the influence of the damping term on the size of that area is currently in progress.

A first step towards confirming the validity of results will be the investigation of the dynamic behaviour associated with the immediately higher order non-linear model. That should allow elicitation of the liquid surface dynamics and the associated stability properties without severe limitations on the excitation amplitude or frequency. Areas of secondary resonance (higher excitation frequency) and also the chaotic areas (higher excitation amplitude) discussed by Ibrahim (2005) are also very interesting topics of further research. These could not be considered here, partly due to time limitations and partly due to limitations of the model.

Of course, experimental reproduction of the identified types of numerical solutions will be required before these are considered as established patterns of the behaviour of the physical system under consideration.

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